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Laplace's equation in polar coordinates examples

In electroquasistatic field problems where the boundary conditions are indicated on circular cylinders or on planes of constant , it is convenient to match these conditions with solutions to Laplace's equation in polar coordinates (cylindrical coordinates without z dependency). The procedure chosen is completely similar to that used in harness 5.4 for Cartesian coordinates. Figure 5.7.1 Polar coordinate system. As a reminder, the polar coordinates are defined in Figure 5.7.1. In these coordinates and with the understanding that there is no z dependency, Laplace's equation, Table I, (8), a difference between this equation and the Laplace equation written in Cartesian coordinates is immediately obvious: In polar coordinates, the equation contains coefficients that not only depend on the independent variable r but become singular by the origin. This unique behavior of differential equation will affect the type of solutions we are now achieving. To reduce the solution of partial differential equation to the simpler problem of solving total differential equations, we are looking for solutions that can be written as products of functions of r alone and alone. When this assumed form is introduced in (1) and the result divided by and multiplied by r, we get We find on the left side of (3) a function of r alone and on the right side a function of alone. The two sides of the equation can balance if and only if the function of and function of r are both equal to the same constant. For this separation constantly we introduce the symbol -m2. For m2 > 0, the solutions for differential equation for F are conveniently written as Due to the room-axing coefficients, the resolutions to (5) are not exponential or linear combinations of exponential, as has been the case to date. Fortunately, the solutions are nonetheless simple. Substitution of a solution with form rn in (5) shows that the equation is met, provided that n = m. Thus, in the specific case of a zero-paring constant, the limiting solutions and the product solutions shown in the first two columns of Table 5.7.1 are constructed by taking all possible combinations of these solutions, those most commonly used in polar coordinates. But what are the solutions if M2 < 0? In Cartesian coordinates, changing the character of the separation constant k2 is similar to switching roles x and y coordinates. Solutions that are periodic in the x direction become exponential, while the exponential decay and growth of the y-direction becomes intermittent. Here the geometry is such that r and coordinates are not interchangeable, but the new solutions resulting from the replacement m2 of -p2, where p is a real number, mainly make oscillating dependency radial instead of azimuthal, and exponential dependence azimuthal rather Radial. To see this, let m2 = -p2, or m = jp, and the solutions that (7) become These take a more familiar look, if it is recognized that r can be written alike as Introducing this identity in (10) then provides the more familiar complex exponentials, which can be divided into its real and imaginary parts using Euler's formula. Two independent solutions for R(s) are thus due to In r's cosine and sine functions. The dependency is now either represented by \exp p or hyperbolic functions that are linear combinations of these exponential. These solutions are summarized in the right column of Table 5.7.1. In principle, the solution to a given problem can be addressed by methodically removing solutions from the catalogue in Table 5.7.1. In fact, most problems are best addressed by attributing some physical importance to each solution. This makes it possible to define coordinates so that field representation is kept as simple as possible. To this end, first consider the solutions found in the first column of Table 5.7.1. The constant potential is an obvious solution and does not need to be considered further. We have a solution in row two where the potential is proportional to the angle. The equation lines and field lines are illustrated in figure 5.7.2a. Evaluating the field by taking the gradient of the potential of polar coordinates (the gradient operator in Table I) shows that it becomes infinitely large as the origin is reached. The potential increases from zero to 2 as the angle increases from zero to 2. If the potential is to be valued simply, we cannot allow this increase to increase further without leaving the region for the validity of the solution. This observation identifies the solution with a physical field observed when two semi-infinite conductive plates are held at different potentials, and the distance between the conductive plates at the junction is assumed to be negligible. In this case, shown in figure 5.7.2, the outside between the plates is correctly represented by a potential proportional to . With the plates separated by an angle of 90 degrees instead of 360 degrees, the potential proportional to the corners of the configuration shown in Figure 5.5.3 is shown. The M2 = 0 resolution in the third row is well known from sec. The fourth m2 = 0 resolution is outlined in figure 5.7.3. To outline the potentials corresponding to the solutions in the second column of Table 5.7.1, the separation constant must be specified. Let us assume at the moment that m is an integer. For m = 1, the solutions r cos and r sin represent familiar potentials. Please note that the polar coordinates are related to the Cartesian coordinates defined in Figure 5.7.1 in Figure 5.7.2 Axe and field lines for a) = , b) area outside the Figure 5.7.3 Equipotentials and field lines for = , In (s). The fields that go with these potentials are best found by taking gradient in Cartesian coordinates. This makes it clear that they can be used to represent consistent fields with x and y directions respectively. In order to emphasise the simplicity of these solutions, which are complicated by polar representation, the second function (13) is shown in Figure 5.7.4a. Figure 5.7.4 Equivalency well edge and field lines for a) = r sin), b) = r-1 , sin (). Figure 5.7.4b shows the potential r-1 sin. To stay at a contour of constant potential in the first quadrant of this figure, which is increased mod / 2, it is necessary first to increase r, and then as the sinus function falls in the second quadrant, to decrease r. The potential is singular by the origin of r; whereas, since the origin is approached from above, it is large and positive; whereas from below it is large and negative. Thus, the field lines emerge from the origin within 0 < < and converge toward the origin in the lower half-plane. There must be a source of origin that consists of equal and opposite charges on the two sides of the planet r sin = 0. The source, which is uniform and of infinite scope in z direction, is a line dipole. This conclusion is confirmed by a direct assessment of the potential generated by two line taxes, the tax -l situated at the origin, the tax +l at a very small distance from the origin at r = d, = /2. The potential arises from steps parallel to those used for the three-dimensional dipole in sec. The spatial dependence on potential is actually sin/r. In an analogy with the three-dimensional dipole of sec. 4.4, p\equiv l d is defined as line-dipole-lied moment. Example 4.6.3 demonstrates that parallel line charges equine the equine are circular cylinders. As this result is independent of the distance between the line charges, it is no surprise that equineotentials of fig. 5.7.4b are circular. In summary, m = 1 solutions can be perceived as the areas of dipoles by infinity and by origin. For its dependencies, the dipoles y are fixed, while for cosin dependencies they are x fixed. The solution of fig. 5.7.5a, \propto r2 its 2, has previously been met in Cartesian coordinates. Either from a comparison of the equivalency parcels or by direct conversion of the Cartesian coordinates into polar coordinates, the potential is recognized as xy. Figure 5.7.5 Equino good edge and field lines for a) = r2 sin (2), b) = r-2 sin (2). M = 2 solution singular at the time of origin is shown in Figure 5.7.5b. Field lines come out of origin and return to the double as range from 0 to 2. This observation identifies four line charges of the same order of magnitude that are alternately drawn as the source of the field. Thus, m = 2 solutions can be regarded as quadruple by infinity and origin. It is perhaps a little surprising that we have gotten from Laplace's equation solutions that are singular on origin and thus associated with sources of origin. The singularity of one of the two independent solutions for (5) can be traced back to the singularity of the coefficients of this differential equation. It is apparent from the above that an increasing m introduces a faster variation of the field in terms of angle coordinate. In trouble where the area of interest includes all values of , m must be an integer to make the field back to the same value after a revolution. But M doesn't have to be an integer. If the area of interest is pie shaped, m can be selected so that the potential passes through a cycle over any interval of . For example, the periodicity angle can be made o by making mo = n or m = n /o, where n can have an integer value. Figure 5.7.6 Equine burns and field lines representative of solutions in the right column of Table 5.7.1. The potential is given by (15). The solutions for m2 < 0, the right column in Table 5.7.1, are illustrated in figure 5.7.6, which mainly uses the fourth resolution as an example. Note that the radial phase has been switched by subtracting p In (b) from the sinus argument. Thus, the potential shown and it passes automatically through zero at radius r = b. The distances between radii with zero potential are not the same. Nevertheless, the potential allocation is qualitatively equivalent to that indicated in Figure 5.4.2. The exponential addiction is azimuthal; this direction thus corresponds to y in Figure 5.4.2. In essence, the potentials for m2 < 0 are similar to those in Cartesian coordinates, but wrapped around the z axis. Page 2 An approach to solving Poisson's equation in a region bounded by surfaces with known potential was outlined in sec. The potential was divided into a specific part if the laplacian balances - /o throughout the region of interest and a homogeneous part that causes the sum of the two potentials to meet the boundary conditions. In short, and on the enveloping surfaces, the following examples illustrate this approach. At the same time, they demonstrate the use of the Cartesian coordinate solutions for Laplace's equation and the idea that the fields described can be time-typical. The cross-section of a two-dimensional system that extends to infinity in the x and z direction is shown in Figure 5.6.1. Leaders in aircraft y = a and y = -a bound region of interest. Between these planes, the charging density is periodically in the x direction and evenly distributed in the y direction. Figure 5.6.1 Cross-section of layers of charge that are periodically in x direction and bounded from above and below by zero potential plates. With this charge to the right, an isolated electrode inserted into the lower equitonic is used to detect the movement. The parameters o and are given constants. For now, the segment connected to the ground through the resistance of the lower electrode can be considered to be at the same zero potential as the rest of the electrode in the plane x = -a and the electrode in the plane y = a. First, we ask for the field distribution. Remember that a specific solution for (2) will do. Since the charging density is independent of y, it is natural to look for a specific solution with the same property. Then left in (2) is another derivative in terms of x, and the equation can be integrated twice to achieve This particular solution is independent of y. Note that this is not the potential that would be achieved by evaluating superposition integrated over the charge between the grounding aircraft. Seen all over the room, that tax distribution is not independent of y. In fact, the potential of (6) is associated with a tax distribution as indicated by (5) that extends to infinity in the +y and -y direction. The homogeneous solution should make it clear that (6) does not meet the limit conditions. That is, at the limits = 0 in (1), so the homogeneous and specific solutions must be balanced there. Thus, we are looking for a solution to Laplace's equation, (3) that meets these border conditions. Since the potential has the same value on the borders, and the origin of the y axis is chosen to be midway between, it is clear that the potential must be an even function of y. In addition, it must have a frequency in the x-direction corresponding to (7). Thus, from the list of solutions to Laplace's equation in Cartesian coordinates in the middle column of Table 5.4.1, k =, its k-x terms are eliminated in favor of cos k-x solutions, and the cosh ky solution is chosen because it is itself in y. The coefficient A is now adjusted so that the limit conditions are met by replacing (8) to (7). The superposition of the solution in question (7) and the homogeneous solution given by replacing the coefficient (9) to (8) result in the desired potential distribution. The mathematical solutions used for diversion (10) are illustrated in Figure 5.6.2. The special solution describes an electric field that originates from areas with positive charge density and ceases in areas with negative charge density. It is purely x directed and is therefore tangential to the equitonic limit. The homogeneous solution added to this field is solely due to surface charges. These give rise to a field that bends out tangential field on the walls, making them surfaces of constant potential. Thus, the sum of the solutions (also shown in the figure) meets Gauss' law and the border conditions. Figure 5.6.2 and field lines for configuration of fig. 5.6.1, graphically showing the overlay of specific and homogeneous parts that provide the necessary potential. With this static view of the fields in mind, assume that the charging distribution moves in the x direction with the speed v. The variable x in (5) has been replaced by x - vt. This transfer tax allocation also moves the field. Thus (10) notes that the homogeneous resolution is now a linear combination of the first and third resolutions in the middle column of Table 5.4.1. As the space charge wave moves by, the charges induced on perfectly conductive walls follow the synchronicity. The current accompanying the redistribution of surface charges shall be recorded if part of the wall is isolated from the rest and connected to the ground through a resistance, as shown in Figure 5.6.1. Under the assumption that the resistance is small enough so that the segment remains at significant zero potential, what is the output voltage vo? The flow through the resistance is found by relying on fee retention for the segment to find the power that is the time change rate of the net charge on the segment. The latter follows from Gauss's integrated law and (12) as it follows that the dynamics of the travelling wave of space loading are reflected in a measured tension in Writing this term, double angle formulas have been invoked. More predictions should be consistent with intuition. The output voltage varies sinus-shaped over time at a frequency proportional to the speed and inversely proportional to the wavelength, 2 /. The higher the speed, the greater the voltage. Finally, if the detection electrode is a multiple of the wavelength 2/, the voltage is zero. If the tax density is concentrated in surface-like areas that are thin compared to other dimensions of interest, it is possible to solve Poisson's equation with border conditions by means of a procedure that has the impression of solving Laplace's equation instead of Poisson's equation. The potential is typically divided into piece-wise continuous functions, and the effect of the charging density is brought in by Gauss' continuity mode, which is used to splice the features on the surface occupied by the charging density. The following example illustrates this procedure. What is achieved is a solution to Poisson's equation throughout the region, including the charging surface. In microwave amplifiers and oscillators of the electron beam type, a fundamental problem is the evaluation of the electrical field produced by a bundled electron beam. The cross-section of the beam is usually small compared to a spare space wavelength of an electromagnetic wave, in which case the electroquassistatic approximation applies. We consider a strip electron beam with a charge density that is uniform above its . The beam moves at speed v in x direction between two planar perfect conductors located on y = a and held at zero potential. The configuration is shown in cross-section in figure 5.6.3. In addition to the uniform charge density, there is a ripple of charge density, so that net charge density is where o, 1, and \ Lambda are constants. The system can be idealized for being infinite in x and y directions. Figure 5.6.3 Cross-section of the plate beam between flat parallel equitonic plates. Beam is modeled on surface charge density with dc and ac parts. The thickness of the beam is much less than the wavelength of the periodic charge density ripple, and much less than the distance 2a of the planar conductors. Thus, the beam is treated as a sheet of surface charge with a density where o = o and 1 = 1 . In regions (a) and (b), respectively above and below the beam, potentially obeying the Laplace equation. Superscript (a) and (b) are now used to specify variables that are evaluated in these areas. To ensure that the basic laws are met in the sheet, these potentials must meet the jumping conditions of the laws of Faraday and Gauss (5.3.4) and (5.3.5). That is, at y = 0 To complete the specification of the field in the area between the plates, the boundary conditions are, on y = a, and on y = -a, In the respective regions, the potential is divided into dc and ac parts, respectively, produced by the uniform and ripple parts of the charging density. By definition, o and 1 meet Laplace's equation and (17), (19) and (20). D.C. part, o, meets (18) with only the first term on the right, while ac del, 1, meets (18) with only the second term. D.C. surface charge density is independent of x, so it's natural to look for potentials that are also independent of x. From the first column of Table 5.4.1, such solutions are the four coefficients in these terms are determined from (17) to (20), if necessary by substitution of these terms and formal solution for the coefficients. More attractive is the solution for inspection, which recognises that the system is symmetrical in terms of y, that the uniform surface charge gives rise to uniform electrical fields directed upwards and downwards in the two regions, and that the associated linear potential must be zero at the two limits. Now consider the ac part of the potential. X addiction is suggested by (18), which makes it clear that for product solutions, x dependence on the potential must be cosine function moves with time. Neither sinh nor cosh functions disappear at the borders, so we will have to take a linear combination of these to meet the border conditions of y= +a. This is done effectively by inspection if it is recognised that the origin of the y axis used in writing is The solutions to Laplace's equation that meet the boundary conditions (19) and (20), these potentials must match at y= 0, as required by (17), so we might as well have written them with the coefficients adjusted accordingly. One remaining coefficient is determined by replacing these terms in (18) (with omitted). We have found the potential as a piecemeal continuous function. In region (a), it is the superposition of (24) and (28), whereas in region (b) it is (25) and (29). In both terms, C is delivered by (30). When t = 0, the ac portion of this potential distribution is as shown by figure 5.6.4. With increasing time, the field distribution translates to the right with the speed v. Note that some lines of electric field intensity that originate on the beam end up elsewhere on the beam, while others end up on the equitonic walls. If the walls are even a wavelength away from the beam (a = \Lambda), almost all field lines end up elsewhere on the beam. That is, coupling to the wall is significant if the wavelength is in the order or greater than one. The nature of solutions to Laplace's equation is in evidence. Two-dimensional potentials, which vary rapidly in one direction, must decay just quickly in a perpendicular direction. Figure 5.6.4 Equial and field lines caused by AC charge in the configuration of fig. 5.6.3. A comparison of the fields from the sheet bar in Figure 5.6.4 and the periodic distribution of volume load density shown in Figure 5.6.2 is a reminder of the similarity between the two physical situations. Although Laplace's equation applies in the subregions of the configuration considered in this section, it is really Poisson's equation that is resolved by and large, as in the previous example. Page 3 Each of the solutions obtained in the previous section by separation of variables can result by appropriate potential applied to pairs of parallel surfaces in the planes x = constant and y = constant. For example, consider the fourth resolution in column k2 0 in Table 5.4.1, which with a constant multiplier is Figure 5.5.1 Two of the infinite number of potential functions in the form (1) that match the limit conditions = 0 at y = 0 and x = 0 and x = a. This resolution has = 0 in planet y = 0 and in planes x = n /k, where n is an integer. Suppose we set k =n/a so = 0 in plan y =a as well. So on y=b, the potential of (1) has a sinus-shaped dependence on x. If a potential in terms of (2) was applied along the surface at y=b, and the surfaces at x= 0, x = a, and y=0 were held at zero potential (with, say, planar conductors held at zero potential), then the potential, (1), would exist within space 0< x < a, 0 < y < b. Segmented electrodes that have each segment limited to the relevant can be used to approximate the distribution at y = b. Potential and field ranges for n = 1 and n = 2 are given in Figure 5.5.1. Note that the statement in sec. 5.2 ensures that the specified potential is unique. But what can be done to describe the field if the wall potentials are not limited to fitting neatly the solution achieved by separating variables? For example, suppose that the fields are desired in the same area of rectangular cross-section, but with an electrode of y = b limited to having a potential v that is independent of x. The configuration is now as shown in figure 5.5.2. Figure 5.5.2 Cross-section of the zero-paw rectangular opening with an electrode with the potential v inserted at the top. A line of attack is proposed by the infinite number of solutions that take the form of (1) that meet the border condition on three of the four walls. The superposition principle makes it clear that any linear combination of these is also a solution, so if we leave an arbitrary coefficient, a more general solution is note that k has been assigned values so that the sine function is zero in planes x = 0 and x = a. Now how can we adjust the coefficients so that the limit condition on the driven electrode, on y = b, is met? An approach that we do not need to use is proposed using the numeric method described in section 4.8. The electrode can be divided into N segments and (3) evaluated in the middle of each segment. If the infinite series were truncated by N-terms, the result would be N equations that were linear in N-unknown An. This system of equations could be reversed to determine An's. If these are replaced by (3), will then include a solution to the limit value problem. Unfortunately, to achieve reasonable accuracy, large values of N would be required and a computer would be needed. The strength of the approach to variable separation is that it results in solutions that are orthogonal in a sense that makes it possible to explicitly determine the coefficients An. The evaluation of the coefficients is remarkably simple. First, (3) is evaluated on the surface of the electrode, where the potential is known. On the right is the infinite series of sinus-shaped functions with coefficients to be determined. On the left is a given function of x. We propagate both sides of the term with sin (m x/a), where m is an integer, and then both sides of the expression are embedded over the width of the system. The functions (n x/a) and sin (m x/a) are orthogonal in the sense that the integrateds of their product above the specified range are zero, unless m = n. Thus, all the terms on the right of (5) disappear, except the one has n =m. Of course, m can be any integer so we can solve (5) for m-th amplitude and then replace m with n. Given the possible distribution of potential on the surface y = b and thus the coefficients that are determined. In this specific problem, the potential is v at each point on the electrode surface. 7 is thus evaluated in order to give final substitution of these coefficients in (3) the desired potential. Each product expression in this infinite series meets Laplace's equation and zero potential condition on three of the surfaces that envelop the region of interest. The sum meets the potential condition on the last limit. Note that the sum itself is not in the form of the product of a function of x alone and a function of y only. The modal enlargement can be used with an arbitrary distribution of the potential on the last limit. But what if we have an arbitrary distribution of potential on all four aircraft enclosing the region of interest? The principle of superposition justifies the use of the sum of four solutions of the type illustrated here. Added to the series solution already found are three more, each similar to the previous one, but rotated by 90 degrees. Since each of the four series has only a limited potential on the part of the limit to which the series applies, the sum of the four meets all limit conditions. The potential of (9) is illustrated in Figure 5.5.3. In the three-dimensional depiction, it is especially clear that the field is infinitely large in the corners, where the driven electrode meets the earthed walls. Where the electric field comes from the powered electrode, there is surface charging, so at the corners there is an infinite surface charge density. In practice, of course, the distance is not vanishingly small and the fields are not infinite. Figure 5.5.3 Potential lines and field lines for the configuration of fig. 5.5.2 (9), shown using vertical coordinates to display the potential and shown in x - y plane. Demonstration 5.5.1. Capacitance attenuator Since none of the field legislation in this chapter involves time derivatives, the area that has been determined is correct for v = v(t), an arbitrary function of time. As a consequence, coefficients An are also functions of time. Thus, the charges induced on the walls of the box are time varying, as can be seen if the wall on y = 0 is insulated from ground side walls and connected to the ground through a resistance. The configuration is displayed in cross-section of figure 5.5.4. The resistance R is small enough that the potential vo is small compared to v. Figure 5.5.4 The bottom of the opening is replaced by an insulating electrode connected to the ground through a low resistance so that the induced current can be measured. The charge induced on this output electrode is found by applying Gauss' integrated law with an integration surface enclosing the electrode. The width of the electrode in the z direction is w, so this expression is evaluated using (9). The conservation of the tax requires that the flow through the rate of change of this tax in terms of time. Thus, the output voltage is, and if v = V sin t, then the experiment shown in Figure 5.5.5 is designed to demonstrate the dependence of the output voltage on the distance b between input and output electrodes. It follows from (13) and (11) that this voltage can be written in normalized form as Figure 5.5.5 Detection of electroquasistatic attenuator, where normalized output voltage is measured as a function of the distance between the input and outputlects normalized to the smaller dimension of the box. The normalization voltage U is defined by (14). The output electrode is placed using the attached insulating rod. In operation, a metal lid covers the side of the box Thus, the natural log of the normalized voltage is the dependency on the electrode distance shown in figure 5.5.5. Note that with increasing b/a the feature quickly becomes a straight line. In the boundary of large b/a, the hyperbolic sinus can be approximated by exp (n b/a)/2 and the series can be approximated with an expression. Thus, dependence on the output voltage on electrode distance becomes simply, and so the asymptotic slope of the curve is -. Charges induced on the input electrode have their images either on the side walls of the box or on the output electrode. If b/a is small, almost all of these images are on the output electrode, but as it is retracted, more and more of the images are on the side walls and fewer are on the output electrode. In retrospect, there are several issues that deserve further discussion. Firstly, the potential used as a starting point in this section, point 1, is one from a list of four in Table 5.4.1. What type of procedure can be used to select the appropriate form? In general, the resolution used to meet zero potential boundary state on the first three surfaces is a linear combination of the four possible solutions. Thus, with A's densifying indeterminate coefficients, the general form of the solution is formal, (1) was chosen by removing three of these four coefficients. The first two must disappear because the function must be zero at x = 0. The third is excluded because the potential must be zero at y = 0. Thus we are led to the last parliamentary term which, if A4 = A, is (1). The methodological elimination of solutions is necessary. Because the coordinates are arbitrary, it is a matter of choosing the origin of the coordinates correctly so that as many of the solutions (16) are eliminated as possible. We deliberately select the origin so that a single expression from the four in (16) meets the limit condition at x = 0 and y = 0. The selection of product solutions from the list should interact with the choice of coordinates. Some combinations are much more convenient than others. This will be exemplified in this and the following The rest of this section is devoted to a more detailed discussion of the enlargement of the Sinusoids represented by (9). In plan y = b, the potential distribution of the form in which the procedure for determining the coefficients has led to (8) is written here in the form of the coefficients Vn (17) as the approximation to the potential v, which is uniformly above the span of the drivelectrod, is shown in Figure 5.5.6. Equation (17) represents a square wave of period 2a spanning all x, - < x < +. Half of a period appears as shown in the shape. It is possible to represent this distribution in terms of sinusoids alone, because it is strange in x. In general, a periodic function is represented by a Fourier series of both sinus and cosines. In the current problem, cosines were missing because the potential had to be zero at x = 0 and x = a. Examination of a Fourier series shows that the series converges to the actual function in the sense that in the sense that in the limit of an infinite number of expressions where (x) is the actual potential distribution and F (x) the Fourier series is approximation. Figure 5.5.6 Fourier series approximation to square wave given by (17) and (18), successively showing one, two and three expressions. Higher order conditions tend to fill in the sharp discontinuity at x = 0 and x = a. Outside the range of interest, the series represents an odd function x with a periodicity length 2a. To see the general approach exemplified here, we show that orthogonality property features X (x) are due to differential equation and boundary conditions. It should therefore not be surprising that the solutions in other coordinate systems also have an orthogonal property. In all cases, the orthogonal property is associated with one of the factors in a product solution. For the Cartesian problem dealt with here, it is X(x) that meets the boundary conditions at two points in the room. This is ensured by adjusting the eigenvalue kn = n /a so that eigenfunction or fashion, sin (n x/a), is zero at x = 0 and x = a. This function meets (5.4.4) and the limit conditions. The lowered m is used to recognize that there are an infinite number of solutions to this problem. Another solution, says n-th, must also meet this equation and the border conditions. The orthogonal property of these conditions, utilized in the assessment of the coefficients in the series expansion, is to prove this condition in general, we multiply (20) with Xn and integrate between the points where the boundary conditions apply. By identifying u = Xn and v = dXm/dx, the first expression is integrated with parts to achieve the first expression on the right disappears due to the boundary conditions. Thus (23) if the same steps are completed with n and m are replaced, the result is (25) with n and m replaced. As the first period of (25) is the same as its counterparty this second equation, subtraction of the two expression yields Thus the functions are orthogonal provided to kn km. For this specific problem, eigenfunctions are Xn = sin (n/a) and eigenvalues are kn =n/a. In general, however, we can expect that our product solutions for Laplace's equation in other coordinate systems will result in a number of functions with similar orthogonal properties. Page 4 After examining some general characteristics of solutions to Poisson's equation, it is now appropriate to examine specific methods for solving Laplace's equation subject to limit conditions. Exemplified by this and the next section are three standard steps often used to represent EQS fields. First, Laplace's equation is set up in the coordinate system, where the interfaces are coordinate surfaces. Then, partial differential equation is reduced to a set of common differential equations when separating variables. In this way, an infinite set of solutions is generated. Finally, the border conditions are met by overlaying the solutions available for the separation of variables. This section derives solutions that are natural if border conditions are specified along the coordinate surfaces of a Cartesian coordinate system. It is assumed that the fields depend only on two coordinates, x and y, so that Laplace's equation is (Table I) This is a partial differential equation in two independent variables. A time-honored method of mathematics is to reduce a new problem to a problem that has previously been solved. This reduces the process of finding solutions to the partial differential equation to find solutions to common differential equations. This is achieved by the method of separating variables. It consists in assuming solutions with the special space dependency I (2), X is assumed to be a function of x alone and Y is a function of y only. If necessary, a general space dependency is regained by overlaying these special solutions. Substitution of (2) in (1) and division before then gives Total derivative symbols used because the respective functions X and Y by definition are only functions x and y. I (3) we now have on the left side a function of x alone, on the right side a function of y alone. The equation can only be met independently of x and y if each of these terms is constant. We denote this separation constant with k2, and it follows, and these equations have the solutions If k = 0, degenerates the solutions for the Product Solutions (2), summarized in the first four rows of Table 5.4.1. The ones in the right column are simply those of the middle column with roles x and y replaced. In general, we will let prime off k' writing these solutions. Exponentials are also solutions to (7). These, sometimes more convenient, solutions are in the last four rows of the table. The solutions summarized in this table can be used to gain insight into the nature of EQS fields. A good investment is therefore made if they are now visualised. The fields represented by the potentials in the left column of Table 5.4.1 are all familiar. Those that are linear in x and y represent consistent fields, in x and y directions, respectively. The potential xy is well known from fig. 4.1.3. We will use similar conventions to represent the potential of the second column, but it is useful to have in mind the three-dimensional depiction exemplified for the potential xy in fig. 4.1.4. In the more complicated field maps to follow, the sketch is visualized as a contour map of the potential with peaks of positive potential and valleys with negative potential. On the top and left periphery of fig. 5.4.1, the functions cos k-x and cosh ky, the product of which is the first of the potentials of the middle column in Table 5.4.1, is outlined. If we start from the origin in either +y or -y directions (north or south), we climb a potential hill. If we instead continue in +x or -x directions (east or west), we move downhill. An easterly path begun on the potential hill north of the origin corresponds to a decrease in cos k-x factor. To follow a path of equal height, the cosh ky factor must be increased, which means that the path must turn north. Figure 5.4.1 Equinos for = cos(kx) cosh (ky) and field lines. To help visualize the potential, the separate factors cos (k-x) and cosh (ky) are displayed at the top and left respectively. A good starting point in making these field sketches is the identification of the contours of zero potential. In the plot of second potential in the middle column of Table 5.4.1, shown in Figure 5.4.2, these y-axis and lines are k-x = + /2, + 3/2, etc. The dependence on y is now strange rather than equal, as it was for the plot of Fig. 5.4.1. Thus, the origin is now on the side of a potential hill that slopes downwards from north to south. Figure 5.4.2 Equinia for = cos (k-x) sinh (ky) and field lines. To help visualize the potential, the separate factors cos (k-x) and sinh (ky) appear respectively at the top and left. The solutions in the third and fourth row of the second column have the same field patterns as those that have just been discussed, provided that these patterns are moved in the x direction. In the last four rows of Table 5.4.1, there are four additional possible solutions, which are linear combinations of the previous four in that column. Because these decays exponentially in either +y or -y directions, they are useful for representing solutions in problems where an infinite half space is considered. The solutions in Table 5.4.1 are non-vascular throughout the x-y aircraft. This that Laplace's equation is obeyed throughout the finite x-y plane, and thus the field lines are continuous; they do not appear or disappear. The sketches show that the fields become stronger and stronger as the positive and negative directions continue. The lines of electric field strain on positive charge and terminate on negative charges on y . Thus, for the parcels shown in Figures 5.4.1 and 5.4.2, the distribution of the tax shall consist of alternating distributions of positive and negative chargeloads of infinite amplitude. Two final observations serve to further develop an understanding of the nature of solutions to Laplace's equation. First, the third dimension can be used to represent the potential in the same way as Fig. 4.1.4, so that the potential surface has the shape of a membrane stretched from boundaries that is elevated in relation to their potentials. Laplace's equation, (1), requires that the sum of quantities that reflect curvatures in x and y directions disappear. If the second derivative of a function is positive, it is curved upwards; and if it is negative, it is curved downwards. If the curvature is positive in the x direction, it must be negative in the y direction. By the origin of fig. 5.4.1, the potential is thus cupped downwards for excursions in the x-direction, and therefore it must be cupped upwards for variations in y direction. A similar deduction shall apply at each point of the x-y-planet. Secondly, because the k that appears in the periodic functions of the second column of Table 5.4.1 is the same as in exponential and hyperbolic functions, it is clear that the faster the periodic variation, the faster the decay or apparent growth. TABLE 5.4.1 TWO-dimensional Cartesian solutions of LAPLACE's EQUATION k = 0 k2 0 k2 0 (k jk') Constant cos kx cosh ky cosh k'x cos k'y cos kx sinh ky coshk'x sin k'y x sin x cosh k 'y xy sin kx sinh sinh k'x sin k'y cos kx eky ek'x cos k'y cos kx e-ky e-k'x cos k'y sin kx eky ek'x sin k'y sin kx e-ky e-k'k'x sin k'y Page 5 On the surfaces of metal conductors, charging dens accumulates, there are only a few atomic distances thick. When describing their fields, the details of the distribution within this thin layer are often not of interest. Thus, the charge is represented by a surface charge density (1.3.11) and the surface supports charge treated as a surface of discontinuity. In such cases, it is often convenient to split a volume in which the field is to be determined in areas separated by the surfaces of discontinuity, and to use equalizing continuous functions to represent the fields. Continuity conditions are then needed to connect field solutions in two regions separated by discontinuity. These conditions are implied by the differential equations that apply throughout the region. They assure that areas are in accordance with the basic laws, even in passing through discontinuity. Each of the four Maxwell equations implies a condition of continuity. Due to the singularity of the source distribution, these laws are used in integrated form to relate the fields to both sides of the surface of discontinuity. Since the vector defined as the unit is normal for the surface of discontinuity and points from area b) to area (a), the continuity conditions are summarized in Table 1.8.3. In EQS approximation, laws of primary interest are Faraday's law without the magnetic induction and Gauss' law, the first two equations of the Chap. 4. Thus, the corresponding EQS continuity conditions are because the magnetic induction does not make any contribution to Faraday continuity mode in any case, these conditions are the same as for the general electrodynamic laws. As a reminder, the outline that encloses the integration surface over which Faraday's Law was integrated (sec. 1.6) to achieve (1) is displayed in Figure 5.3.1a. The integration volume used to obtain (2) from Gauss Law (see 1.3) is also shown in Figure 5.3.1b. Figure 5.3.1 a) Differential contour that cuts surface recharging density. (b) Differential volume enclosing surface charge on the surface with normal n. What are the continuity conditions on electrical potential? The potential is continuously across a surface of discontinuity, although this surface carries a surface charge density. This will be the case when the E-field is final (a dipole layer containing an infinite field, causing a leap of potential), because then the line integrated by the electric field from one side of the surface to the other side is zero, the path-length is infinitely small. To determine the jumping state representing Gauss's law through the surface of discontinuity, it was integrated (see 1.3) above the displayed volume that cuts the surface into Fig. 5.3.1b. The resulting continuity condition (2) is written with regard to the potential of recognizing that E = - in the EQS approximation. On a surface of discontinuity that carries a surface charging density, the normal derivative of potential is discontinuous. The continuity conditions become border conditions if they are designed to represent physical limitations that go beyond those already implied by the laws prevailing in the volume. A well-known example is one where the surface is that of an electrode limited in its potential. Then, the continuity mode (3) requires that the potential of the volume adjacent to the electrode is the given potential of the electrode. This statement cannot be justified without relying on information on the physical nature of the electrode (that it is infinitely conductive, for example) which is not represented in the volume laws and is therefore not inextricably linked to the continuity conditions. Page We're going to show this section states that a potential distribution that obeys Poisson's equation is completely specified in a volume V if the potential is indicated above the surfaces delimiterting that volume. Such a uniqueness phrase is useful for two reasons: (a) It tells us that if we have found such a solution to Poisson's equation, either by mathematical analysis or physical insight, then we have found the only solution; and (b) it tells us which border conditions are suitable for clearly specifying a solution. If there is no charge present in the amount of interest, then the phrase states it unique in solutions to Laplace's equation. Following the redductio ad absurdum method, we assume that the solution is not unique, that there are two solutions, a and b, that meet the same border conditions, and then show that it is impossible. The presumably different solutions a and b must meet Poisson's equation with the same tax distribution and must meet the same border conditions. It follows that with d defined as the difference in the two potentials, d = a - b, A simple argument now shows that the only way d can both satisfy Laplace's equation and be zero on all delimiter surfaces is for it to be zero. First, it is argued that d cannot have a maximum or a minimum at any time within V. Using fig. 5.2.1, the negative is visualized by a field line as it passes through a point of calm. Because the field is magnetic (divergent), such a field line cannot start or stop within V (sec. 2.7). In addition, the field defines a potential (4.1.4). Therefore, as one continues along the field line in the direction of the negative gradient, the potential has to decrease until the field line reaches one of the surfaces Si demarcates V. Similarly in the opposite direction, the potential has to increase until another of the surfaces is reached. Therefore, all maximum and minimum values d (r) must be located on the surfaces. Figure 5.2.1 Field line originating on one part of the demarcation surface and ending another after passing the point calm The potential of difference at any inside point cannot assume a value greater than or less than the largest or smallest value of the potential on the surfaces. But the surfaces themselves are at zero potential. It follows that the difference potential is zero everywhere in V, and that a = b. Therefore, there is only one solution to the limit value problem specified by (1). Page 7 Suppose that we want to analyze an electroquasistatic situation as shown in Figure 5.1.1. A tax allocation (s) is indicated in the part of the room of interest specified by Volume V. This region is bounded by perfect leaders of specified shape and location. Known potentials are applied to these conductors and the enveloping surface, which can be infinite. Figure 5.1.1 The amount of interest be a distribution of the tax density. To illustrate delineated surfaces where potential is limited, n insulated surfaces and an enveloping surface are displayed. In the space between the leaders, the potential function obeys Poisson's equation (5.0.2). A special solution of this equation within the prescribed volume V is given by superposition integrated, (4.5.3). This potential obeys Poisson's equation at each point in volume V. Since we do not evaluate this equation outside volume V, the integration of the sources required in point 10 000 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 This makes it clear that the solution in question is not unique because the addition to the potential of integrating over arbitrary charges outside volume V will only give rise to potential if the laside derivative is zero within Volume V. Is (1) the complete solution? Because it's not unique, the answer should be, surely not. Furthermore, it is clear that there is no built-in information on the location and form of the managers in this solution. Therefore, the electrical field obtained as the negative gradient of the potential p of (1) will generally have a limited tangential component on the electrodes' surfaces. On the other hand, the conductors have surface charge distributions that adjust themselves so that the net field on the conductors' surfaces disappears tangential electrical field components. The distribution of these surface charges is not known from the outset and cannot therefore be included in the integral (1). One way out of this dilemma is as follows: the potential distribution we seek within the space not occupied by the leaders is the result of two tax distributions. Firstly, the prescribed volume charge distribution leads to the potential function is p and, secondly, the charge is distributed on the conductor surfaces. The potential function produced by surface charges must obey source-free Poisson's equation in room V of interest. Let us describe this solution to the homogeneous form of Poisson's equation by the potential function h. Then, in volume V, h must satisfy the Laplace equation. The superposition principle then makes it possible to write the total potential, as the problem of finding the complete field distribution is now reduced to finding a solution so that the net potential (3) has the prescribed potentials we on the surfaces Si. Now p is known and can be evaluated on the surface Si. Evaluation of (3) on Si thus gives that the homogeneous solution is prescribed on the borders Si. Therefore, the determination of an electroquasistatic field with prescribed potentials at the limits is reduced to the solution of Laplace's equation, (2) that meets the boundary condition given by (5). The approach that has been formalised in the another point of view that applies to the limit value problems in the last part of the chapter. The abstract concept of the limit value situation in Figure 5.1.1 is certainly no different from that of Figure 4.6.1. In example 4.6.4, the field shown in Figure 4.6.8 is determined for a point charge next to an equipel charge-neutral spherical electrode. In volume V of interest outside the electrode, volume charge distribution is singular, the point charge q. Indeed, the potential of (4.6.35) is in the form of (3). The special solution can be taken as the first term, the potential for a point charge. The second and third terms, which correspond to the possibilities caused by the fictitious charges in the field, can be considered as the homogeneous solution. Superposition to meet limit conditions In the following sections, superposition will often be used in a different way to meet the limit conditions. Suppose that there is no tax density in volume V, and again the potentials on each of n surfaces Sj is vj. So the solution is divided into a superposition of solutions j that meet the required mode on the j-th surface, but is zero on all the others. Each expression is a solution to Laplace's equation, (6), so the sum is so good. Viewer 5.5 develops a method of meeting arbitrary border conditions on one of four surfaces that envelops a lot of interest. Capacitance Matrix Assume that in n electrode system net charge on i-th electrode is to be found. Considering (8), the integrating of E da above the surface Si encloses this electrode then provides due to the linearity of the Laplace equation, the potential j is proportional to the voltage exciting this potential, vj. It follows that (11) can be written in the form of capacitance parameters independent of the excitations. That is, (11) becomes where the capacitance coefficients are the charge on the i-th electrode is a linear superposition of the contributions from all n-voltages. The coefficient that multiplies its own tension, Cii, is called selfcapacitance, while the others, Cij, in j, are the mutual capacitances. Page 8 The electroquassist laws were discussed in chapter 1. The electric field intensity E is irrotational and represented by the negative gradient of the electrical potential. Gauss' law is then fulfilled if the electrical potential is related to the tax density by Poisson's equation In charge-free regions of space, obeys the Laplace equation, (2), with = 0. The last part of the Cape. 4 was devoted to an opportunistic approach to finding limit value solutions. An exception was the numerical schema described in section 4.8, which led to the resolution of a limit value problem using the source superposition method. This chapter makes a more direct attack on the resolution of limit value problems without necessarily numeric methods. It is one that will be used extensively not only as effects of polarization and conduction added to EQS laws, but in conjunction with MQS systems as well. Once again, there is an analogy useful for those familiar with the description of linear circuit dynamics in the form of ordinary differential equations. Over time as the independent variable, the answer to a drive that turns on when t = 0 can be determined in two ways. The first represents the answer as a superposition of impulse reactions. The resulting convolution integral represents the answer for all time, before and after t = 0 and even when t = 0. This is the analogy of the point of view that was taken in the first part of the chapter. The second approach represents the history of dynamics before, when t = 0 with respect to original conditions. With the understanding that interest is limited to times after t = 0, the answer is then divided into special and homogeneous parts. The specific solution to the differential equation representing the circuit is not unique, but ensures that the differential equation at every moment in the temporal area of interest is met. This particular solution does not have to meet the original conditions. In this chapter, the tax density is driven, and the particular potential response guarantees that Poisson's equation is met throughout the region of geographical interest. The circuitanaloge uses the homogeneous solution to meet the original conditions. In the field problem, the homogeneous solution is used to meet border conditions. In a circuit, the homogeneous resolution can be perceived as the reaction to drives that occurred before t = 0 (outside the temporal range of interest). In determining the potential distribution, the homogeneous response is a response foreseen in Laplace's equation (2), with = 0, and can be considered either as caused by fictitious taxes residing outside the area of interest or due to the surface taxes associated with the borders. The development of these ideas in Sec 5.1-5.3 is independent and does not depend on a knowledge of circuit theory. However, for those familiar with the solution of common differential equations, it is satisfactory to see that the methods used here for the treatment of partial differential equations are a natural extension of those used for common differential equations. Although it can often be found more simply by other methods, a particular solution always follows from the superposition integrated. The main content of this chapter is therefore to find a determination of homogeneous solutions, to find solutions to Laplace's equation. Many convenient configurations have limits that are described by specifying one of the coordinate variables in a three-dimensional coordinate system equal to a constant. For example, a box with rectangular cross section has walls described by setting a Cartesian coordinate equal to a constant to describe the boundary. Similarly, the boundaries of a circular cylinder are naturally described in cylindrical coordinates. So it is that there is great interest in having solutions to Laplace's equation that naturally fit these configurations. With many examples interwoven into the discussion, much of this chapter is devoted to cataloguing these solutions. The results are used in this chapter to describe EQS fields in free space. However, as the effects of polarization and conduction are added to EQS's responsibilities, and as MQS systems with magnetization and wiring are considered, the homogeneous solutions to the Laplace equation established in this chapter will be a continuous resource. A review of the Cape. 4 will identify many solutions to Laplace's equation. As long as the field source is outside the region of interest, the resulting potential obeys Laplace's equation. What is different about the solutions set out in this chapter? A hint comes from the numerical procedure used in sec. 4.8 to meet arbitrary limit conditions. A superposition of N solutions to Laplace's equation was used to meet the conditions at N-points on the borders. Unfortunately, to determine the amplitudes of these N-solutions, N equations had to be solved for N unknowns. The solutions to Laplace's equation found in this chapter can also be used as expressions in an infinite series designed to meet arbitrary boundary conditions. But what's different about the terms of this series is their orthogonality. This characteristic of the solutions makes it possible to explicitly determine the individual amplitudes in the series. The concept of orthogonality of functions may already be well known through an exposure to Fourier analysis. In any case, the basic ideas are introduced in section 5.5. Page 95.0 Introduction 5.1 Special and Homogeneous Solutions to Poisson's Laplace's Equations 5.2 Uniqueness of Solutions to Poisson's Equation 5.3 Continuity Conditions 5.4 Solutions to Laplace's Equation in Cartesian Coordinates 5.5 Modal Expansion to meet border conditions 5.4 Solutions to Laplace's Equation in Cartesian Coordinates 5.5 Modal Expansion to meet boundary conditions 5.4 Solutions to Laplace's Equation in Cartesian Coordinates 5.5 Modal Expansion to meet border conditions 5.4 Solutions to Laplace's Equation in Cartesian Coordinates 5.5 Modal Expansion to meet border conditions 5.4 Solutions to Laplace's Equation in Cartesian Coordinates 5.5 Modal Expansion to meet boundary conditions 5.4 Solutions to Laplace's Equation in Cartesian Coordinates 5.5 Modal Expansion to meet border conditions 5.4 Solutions to Laplace's Equation in Cartesian Coordinates 5.5 Modal Expansion to meet border conditions 6 Solutions to Poisson's with BoundaryConditions 5.7 Solutions for Laplace's Equation in Polar Coordinates 5.8 Examples in Polar Coordinates 5.9 Three Solutions to Laplace's Equation inSpheros Coordinates 5.10 Three-Dimensional Solutions to Laplace's Equation 5.11 Summary References Problems